

# Non-Abelian BPS Monopoles in N=4 Gauged Supergravity

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We study static, spherically symmetric, and purely magnetic solutions of  $SU(2) \times SU(2)$  gauge supergravity in four dimensions. A systematic analysis of the supersymmetry conditions reveals solutions which preserve 1/8 of the supersymmetries and are characterized by a BPS-monopole-type gauge field and a globally hyperbolic, everywhere regular geometry. These present the first known example of non-Abelian backgrounds in gauge supergravity and in leading order effective string theory.

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**Introduction.**— In the last few years there has been considerable interest in supersymmetric solitons originating from effective field theories of superstrings and heterotic strings (see [1] for review). These solutions play an important role in the study of the non-perturbative sector of string theory and in understanding string dualities. A characteristic feature of such solutions is that supersymmetry is only partially broken, and associated with each of the unbroken supersymmetries there is a Killing spinor fulfilling a set of linear differential constraints. The corresponding integrability conditions can be formulated as a set of non-linear Bogomolny equations for the solitonic background, which can often be solved analytically.

The analysis of the supersymmetry conditions has proven to be the efficient way of studying the non-perturbative sector. So far, however, the investigations have mainly been restricted to the Abelian theory, whereas little is known about supergravity solitons with non-Abelian gauge fields, which presumably is due to the complexity of the problem. The known solutions appear to be somewhat special, since they are constructed either from the flat space configurations of the Yang-Mills field, by making use of the conformal invariance (see [1]–[3] and references therein), or from the gravitating Abelian solitons, by identifying gravitational and gauge connections [4]. At the same time, the example of the well-known (non-supersymmetric) Bartnik-McKinnon particles [5] shows that the generic behavior of a gravitating Yang-Mills field can be far more complex.

Motivated by this, we study solitons in a four-dimensional supergravity model with non-Abelian Yang-Mills multiplets. The model we consider is the N=4 gauged  $SU(2) \times SU(2)$  supergravity [6], which can be regarded as N=1, d=10 supergravity compactified on the group manifold  $S^3 \times S^3$ . The non-gauged version of the same model, corresponding to the toroidal compactification of ten-dimensional supergravity, has been extensively studied in the past [8]. We investigate static, spherically symmetric, purely magnetic field configurations and find in this case analytically *all* supersymmetric solutions. Among them we discover globally regular solutions characterized by a BPS-monopole-type gauge field. The corresponding geometry is globally hyperbolic and does not belong to any standard type. It is worth noting that, although the Abelian solutions in the model were studied long ago [8], to our knowledge, we present here the first example of non-Abelian backgrounds. At the same time, these are the first non-Abelian solutions of the *leading order* equations of motion of the effective string action. All other known solutions [1]–[4] have gauge fields which originate from string corrections.

**The model.**— The action of the N=4 gauged  $SU(2) \times SU(2)$  supergravity theory includes a vierbein  $e_\mu^m$ , four Majorana spin-3/2 fields  $\psi_\mu \equiv \psi_\mu^I$  ( $I = 1, \dots, 4$ ), vector and pseudovector non-Abelian gauge fields  $A_\mu^a$  and  $B_\mu^a$  with independent gauge coupling constants  $e_A$  and  $e_B$ , respectively, four Majorana spin-1/2 fields  $\chi \equiv \chi^I$ , the axion and the dilaton [6]. We consider the truncated theory specified by the conditions  $e_B = B_\nu^a = 0$ . In addition, we require the vector field  $A_\mu^a$  to be purely magnetic, which allows us to set the axion to zero. After a suitable rescaling of the fields, the bosonic part of the action reads

$$S = \int \left( -\frac{1}{4} R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{2\phi} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{8} e^{-2\phi} \right) \sqrt{-g} d^4x, \quad (1)$$

where  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \varepsilon_{abc} A_\mu^b A_\nu^c$ , and the dilaton potential can be viewed as an effective negative, position-dependent cosmological term  $\Lambda(\phi) = -\frac{1}{4} e^{-2\phi}$ .

For a purely bosonic configuration, the supersymmetry transformation laws are [6]

$$\begin{aligned}\delta\bar{\chi} &= -\frac{i}{\sqrt{2}}\bar{\epsilon}\gamma^\mu\partial_\mu\phi - \frac{1}{2}e^\phi\bar{\epsilon}\alpha^a F_{\mu\nu}^a\sigma^{\mu\nu} + \frac{1}{4}e^{-\phi}\bar{\epsilon}, \\ \delta\bar{\psi}_\rho &= \bar{\epsilon}\left(\overleftarrow{\partial}_\rho - \frac{1}{2}\omega_{\rho mn}\sigma^{mn} + \frac{1}{2}\alpha^a A_\rho^a\right) - \frac{1}{2\sqrt{2}}e^\phi\bar{\epsilon}\alpha^a F_{\mu\nu}^a\gamma_\rho\sigma^{\mu\nu} + \frac{i}{4\sqrt{2}}e^{-\phi}\bar{\epsilon}\gamma_\rho,\end{aligned}\quad (2)$$

the variations of the bosonic fields being zero. In these formulas,  $\epsilon \equiv \epsilon^I$  are four Majorana spinor supersymmetry parameters,  $\alpha^a \equiv \alpha_{\mathbf{I}\mathbf{J}}^a$  are the SU(2) gauge group generators, whose explicit form is given in [6], and  $\omega_{\rho mn}$  is the tetrad connection.

We shall consider static, spherically symmetric, purely magnetic configurations of the bosonic fields, and for this we parameterize the fields as follows:

$$\begin{aligned}ds^2 &= N\sigma^2 dt^2 - \frac{dr^2}{N} - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \\ \alpha^a A_\mu^a dx^\mu &= w(-\alpha^2 d\theta + \alpha^1 \sin\theta d\varphi) + \alpha^3 \cos\theta d\varphi,\end{aligned}\quad (3)$$

where  $N$ ,  $\sigma$ ,  $w$ , as well as the dilaton  $\phi$ , are functions of the radial coordinate  $r$ . The field equations, following from the action (1), read

$$\begin{aligned}(rN)' + r^2 N \phi'^2 + U + r^2 \Lambda(\phi) &= 1, \\ (\sigma N r^2 \phi')' &= \sigma(U - r^2 \Lambda(\phi)), \\ r^2 (N \sigma e^{2\phi} w')' &= \sigma e^{2\phi} w(w^2 - 1), \\ \sigma' &= \sigma(r\phi'^2 + 2e^{2\phi} w'^2/r),\end{aligned}\quad (4)$$

where  $U = 2e^{2\phi}(Nw'^2 + (w^2 - 1)^2/2r^2)$ . Now, since we are unable to directly solve these equations, we shall consider the supersymmetry conditions for the fields (3), which will give us a set of first integrals for the system (4).

**The supersymmetry conditions.**— The field configuration (3) is supersymmetric, provided that there are non-trivial supersymmetry Killing spinors  $\epsilon$  for which the variations of the fermion fields defined by Eqs. (2) vanish. Inserting configuration (3) into Eqs. (2) and putting  $\delta\bar{\chi} = \delta\bar{\psi}_\mu = 0$ , the supersymmetry constraints become a system of equations for the four spinors  $\epsilon^I$ . The procedure which solves these equations is rather involved. For this reason we describe here only the principal steps of the analysis. First, since the background field is static and spherically symmetric, we choose the spinors to be time-independent and classify them with respect to the total angular momentum  $J = L + S + I$ . Since spin  $S$  and isospin  $I$  are half-integer,  $J$  is integer, and we restrict to the  $J = 0$  sector. In this sector half of the 16 independent spinor components vanish (in the special representation chosen), which in effect truncates half of the supersymmetries. Hence, the supersymmetry constraints  $\delta\bar{\chi} = 0$  and  $\delta\bar{\psi}_\mu = 0$  for  $\mu = t, \theta, \varphi$  reduce to four systems of homogeneous algebraic equations for the remaining eight spinor components, each system containing eight equations, while the  $\mu = r$  gravitino constraint becomes a system of radial differential equations.

The consistency of the algebraic constraints requires that the determinants of the corresponding coefficient matrices vanish and that the matrices commute with each other.

These consistency conditions can be expressed by the following relations for the background:

$$N\sigma^2 = e^{2(\phi-\phi_0)}, \quad (5)$$

$$N = \frac{1+w^2}{2} + e^{2\phi} \frac{(w^2-1)^2}{2r^2} + \frac{r^2}{8} e^{-2\phi}, \quad (6)$$

$$r\phi' = \frac{r^2}{8N} e^{-2\phi} \left( 1 - 4e^{4\phi} \frac{(w^2-1)^2}{r^4} \right), \quad (7)$$

$$rw' = -2w \frac{r^2}{8N} e^{-2\phi} \left( 1 + 2e^{2\phi} \frac{w^2-1}{r^2} \right), \quad (8)$$

with constant  $\phi_0$ . Under these conditions, the solution of the algebraic constraints yields  $\epsilon$  in terms of only two independent functions of  $r$ . The remaining differential constraint then uniquely specify these two functions up to two integration constants, which finally corresponds to two unbroken supersymmetries. We therefore conclude that the supersymmetry conditions for the bosonic background (3) are given in terms of Eqs. (5)–(8). Solutions of these Bogomolny equations describe the BPS states in N=4 gauge supergravity with 1/8 of the supersymmetries preserved (since the maximal possible number of supersymmetries in the model is 16). One can verify that the Bogomolny equations are compatible with the field equations (4).

**The solution.**— In order to find the general solution of the Bogomolny equations, we start from the case where  $w(r)$  is constant. The only possibilities are  $w(r) = \pm 1$  or  $w(r) = 0$ . For  $w(r) = \pm 1$  the Yang-Mills field is a pure gauge, and the equations imply that  $\exp(-2\phi) = 0$ , which means that  $\phi(r) = \phi_0 \rightarrow \infty$ , whereas the metric is flat. The  $w(r) = 0$  choice corresponds to the Dirac monopole gauge field. The general solution of the remaining non-trivial Eq. (7) is then given by  $\phi + \ln(r/r_0) = r^2 e^{-2\phi}/4$ , with constant  $r_0$ ; the corresponding metric turns out to be singular at the origin.

Suppose now that  $w(r)$  is not a constant. Introducing the new variables  $x = w^2$  and  $R^2 = \frac{1}{2} r^2 e^{-2\phi}$ , Eqs. (5)–(8) become equivalent to one first order differential equation

$$2xR(R^2 + x - 1) \frac{dR}{dx} + (x+1)R^2 + (x-1)^2 = 0. \quad (9)$$

If  $R(x)$  is known, the radial dependence of the functions,  $x(r)$  and  $R(r)$ , can be determined from (7) or (8). Eq. (9) is solved by the following substitution:

$$x = \rho^2 e^{\xi(\rho)}, \quad R^2 = -\rho \frac{d\xi(\rho)}{d\rho} - \rho^2 e^{\xi(\rho)} - 1, \quad (10)$$

where  $\xi(\rho)$  is obtained from

$$\frac{d^2 \xi(\rho)}{d\rho^2} = 2 e^{\xi(\rho)}. \quad (11)$$

The most general (up to reparametrizations) solution of this equation which ensures that  $R^2 > 0$  is  $\xi(\rho) = -2 \ln \sinh(\rho - \rho_0)$ . This gives us the general solution of Eqs. (5)–(8). The metric is non-singular at the origin if only  $\rho_0 = 0$ , in which case

$$R^2(\rho) = 2\rho \coth \rho - \frac{\rho^2}{\sinh^2 \rho} - 1, \quad (12)$$

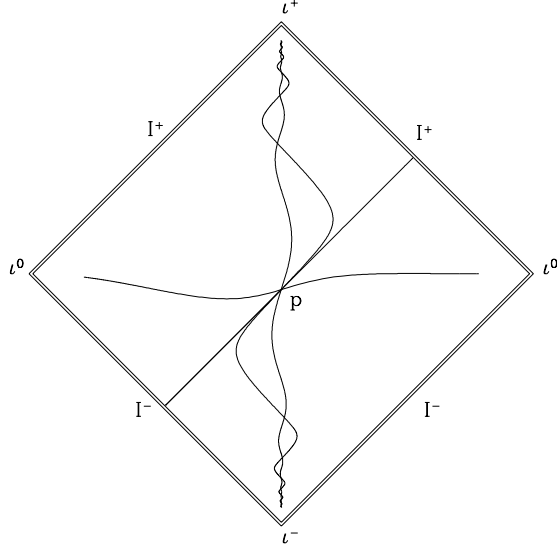


Figure 1: The conformal diagram for the spacetime described by the line element (13).

one has  $R^2(\rho) = \rho^2 + O(\rho^4)$  as  $\rho \rightarrow 0$ , and  $R^2(\rho) = 2\rho + O(1)$  as  $\rho \rightarrow \infty$ . The last step is to obtain  $r(s)$  from Eq. (8), which finally gives us a family of completely regular solutions of the Bogomolny equations:

$$ds^2 = a^2 \frac{\sinh \rho}{R(\rho)} \left\{ dt^2 - d\rho^2 - R^2(\rho)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right\}, \quad (13)$$

$$w = \pm \frac{\rho}{\sinh \rho}, \quad e^{2\phi} = a^2 \frac{\sinh \rho}{2 R(\rho)}, \quad (14)$$

where  $0 \leq \rho < \infty$ ,  $R(\rho)$  is given by Eq. (12), and we have chosen in Eq. (5)  $2\phi_0 = -\ln 2$ . The appearance of the free parameter  $a$  in the solution reflects the scaling symmetry of Eqs. (5)–(8):  $r \rightarrow ar$ ,  $\phi \rightarrow \phi + \ln a$ . The geometry described by the line element (13) is everywhere regular, the coordinates covering the whole space whose topology is  $\mathbb{R}^4$ . The geometry becomes flat at the origin, but asymptotically it is not flat, even though the cosmological term  $\Lambda(\phi)$  vanishes at infinity. We thus can not assign a total energy to the solution. Specifically, in the asymptotic region all curvature invariants tend to zero, however, not fast enough. The Schwarzschild metric functions for  $r \rightarrow \infty$  are  $N \propto \ln r$  and  $N\sigma^2 \propto r^2/4 \ln r$ , the non-vanishing Weyl tensor invariant being  $\Psi_2 \propto -1/6r^2$ .

The global structure of the solution is well illustrated by the conformal diagram. Inspecting the  $t$ – $\rho$  part of the metric, it is not difficult to see that the conformal diagram in this case is actually identical to the one for Minkowski space, even though the geometry is not asymptotically flat (see Fig.1). The spacetime is therefore geodesically complete and globally hyperbolic. The latter property is quite remarkable, since global hyperbolicity is usually lacking for the known supersymmetry backgrounds in gauged

supergravity models. The geodesics through a spacetime point  $p$  are shown in the diagram, each geodesic approaching infinity for large absolute values of the affine parameter. Although the global behavior of geodesics is similar to that for Minkowski space, they locally behave differently. For  $\rho < \infty$  the cosmological term  $\Lambda(\phi)$  is non-zero and negative, thus having the focusing effect on timelike geodesics, which makes them oscillate around the origin. Unlike the situation in the anti-de Sitter case, each geodesic has its own period of oscillations, such that the geodesics from a point  $p$  never refocus again.

The shape of the gauge field amplitude  $w(\rho)$ , given by Eq. (14), corresponds to the gauge field of the regular magnetic monopole type. In fact, replacing  $\rho$  by  $r$ , the amplitude exactly coincides with that for the flat space BPS solution. This result is quite surprising, since the model has no Higgs field, in which case it would be natural to expect the existence of only neutral solutions [5]. Note that all known stringy monopoles in four dimensions [1], [3] contain a Higgs field.

In conclusion, Eqs. (13), (14) describe globally regular, supersymmetric backgrounds of a new type. The existence of unbroken supersymmetries suggests that the configurations should be stable, and we expect that the stability proof can be given along the same lines as in [9]. Being solutions of N=4 quantum supergravity in four dimensions, they presumably receive no quantum corrections. On the other hand, they can be considered in the framework of string theory, and then the issue of string corrections can be addressed. In order to study this problem, we first of all need to lift the solutions to ten dimensions. Although the process is rather involved, one can show that such a lifting is indeed possible.

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